APPROXIMATE SOLUTIONS OF NONSTATIONARY JUNCTION HEAT-EXCHANGE PROBLEMS FOR LAMINAR FLUID FLOW IN CHANNELS

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Through the combined use of the double-integral Laplace–Carson transform and the Bubnov–Galerkin orthogonal method, a solution is obtained of a junction heat-exchange problem for rectilinear fluid flow.

A theoretical basis of the need for solving junction heat-exchange problems was presented in [1] by A. V. Lykov and T. L. Perel'man. P. V. Tsoi [2, 3] developed an approximate analytical method for the study of heat exchange in laminar fluid flow. In the present paper we extend this method to the solution of nonstationary junction heat-exchange problems.

Let two distinct fluids move with identical speeds in plane-parallel channels separated by a thin partition. We neglect the thermal conductivity of the channel walls and of the partition. The fluid flow is laminar. We assume that on one of the outside walls there is no heat exchange, while on the other wall a boundary condition of the first kind is specified. On the contact zone of the two media junction conditions in the form of equality of temperatures and thermal flows are satisfied.

The mathematical statement of the problem is the following:

$$\frac{\partial T_1(\rho,\ z,\ \mathrm{Fo})}{\partial \mathrm{Fo}}+\left(1-\frac{\rho}{\rho_1}\right)\frac{\rho}{\rho_1}\,\frac{\partial T_1(\rho,\ z,\ \mathrm{Fo})}{\partial z}=\frac{a_1}{a}\,\frac{\partial^2 T_1(\rho,\ z,\ \mathrm{Fo})}{\partial \rho^2}, \tag{1}$$

$$\frac{\partial T_2(\rho,\ z,\ \mathrm{Fo})}{\partial \mathrm{Fo}}+\left(1-\frac{\rho-\rho_1}{1-\rho_1}\right)\left(\frac{\rho-\rho_1}{1-\rho_1}\right)\frac{\partial T_2(\rho,\ z,\ \mathrm{Fo})}{\partial z}=\frac{a_2}{a}\,\frac{\partial^2 T_2(\rho,\ z,\ \mathrm{Fo})}{\partial \rho^2}, \tag{2}$$

$$T_i(\rho,\ z,\ 0)=T_{\mathrm{i}\,i}\quad (i=1,\ 2), \tag{3}$$

$$T_i(\rho,\ 0,\ \mathrm{Fo})=T_{0i}\quad (i=1,\ 2), \tag{4}$$

$$\partial T_1(0,\ z,\ \mathrm{Fo})/\partial\rho=0, \tag{5}$$

$$T_1(\rho_1,\ z,\ \mathrm{Fo})=T_2(\rho_1,\ z,\ \mathrm{Fo}), \tag{6}$$

$$\lambda_1\partial T_1(\rho_1,\ z,\ \mathrm{Fo})/\partial\rho=\lambda_2\partial T_2(\rho_1,\ z,\ \mathrm{Fo})/\partial\rho, \tag{7}$$

$$T_2(1,\ z,\ \mathrm{Fo})=T_{\mathrm{w}}\,, \tag{8}$$

where

$$\mathrm{Fo}=\frac{a\tau}{R^2};\quad z=\frac{1}{\mathrm{Pe}}\,\frac{x}{R};\quad \mathrm{Pe}=\frac{6RW_{\mathrm{av}}}{a};\quad W_1(r)=6W_{\mathrm{av}}\left(1-\frac{r}{r_1}\right)\frac{r}{r_1};$$

$$W_2(r)=6W_{\mathrm{av}}\left(1-\frac{r-r_1}{r_2-r_1}\right)\left(\frac{r-r_1}{r_2-r_1}\right).$$

To solve the problem (1)-(8) we employ the double integral Laplace–Carson transform along with the Bubnov–Galerkin orthogonal method.

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We apply the Laplace–Carson double integral transform with respect to the dimensionless time Fo and with respect to the coordinate z. To this end we let

$$\bar{T}^* = ps\int_0^\infty\int_0^\infty T(\rho,\ z,\ \mathrm{Fo})\exp[-(p\,\mathrm{Fo}+sz)]\,d\,\mathrm{Fo}\,dz.$$

In the transform domain we can write problem (1)-(8) as follows:

$$p[\bar{T}_1^*(\rho,\ s,\ p)-T_{i1}]+s\left(\frac{\rho}{\rho_1}-\frac{\rho^2}{\rho_1^2}\right)[\bar{T}_1^*(\rho,\ s,\ p)-T_{01}]-\frac{a_1}{a}\ \frac{\partial^2\bar{T}_1^*(\rho,\ s,\ p)}{\partial\rho^2}=0, \tag{9}$$

$$p[\bar{T}_2^*(\rho,\ s,\ p)-T_{i2}]+s\left[\frac{\rho-\rho_1}{1-\rho_1}-\frac{(\rho-\rho_1)^2}{(1-\rho_1)^2}\right][\bar{T}_2^*(\rho,\ s,\ p)-T_{02}]-\frac{a_2}{a}\ \frac{\partial^2\bar{T}_2^*(\rho,\ s,\ p)}{\partial\rho^2}=0, \tag{10}$$

$$\partial\bar{T}_1^*(0,\ s,\ p)/\partial\rho=0, \tag{11}$$

$$\bar{T}_1^*(\rho_1,\ s,\ p)=\bar{T}_2^*(\rho_1,\ s,\ p), \tag{12}$$

$$\lambda_1\partial\bar{T}_1^*(\rho_1,\ s,\ p)/\partial\rho=\lambda_2\partial\bar{T}_2^*(\rho_1,\ s,\ p), \tag{13}$$

$$\bar{T}_2^*(1,\ s,\ p)=T_w. \tag{14}$$

In accordance with the Bubnov–Galerkin method we apply a first approximation to the problem (9)-(14) in the form

$$\bar{T}_{1i}^*(\rho,\ s,\ p)=T_w+b_1(s,\ p)\varphi_{1i}(\rho)\quad(i=1,\ 2). \tag{15}$$

We take as our coordinate functions the following:

$$\varphi_{11}(\rho)=\left(1-\frac{\lambda_2}{\lambda_1}\right)(1-\rho_1^2)+\frac{\lambda_2}{\lambda_1}(1-\rho^2), \tag{16}$$

$$\varphi_{12}(\rho)=1-\rho^2. \tag{17}$$

When relations (16) and (17) are taken into account, the expression (15) satisfies the boundary conditions (11) and (14) and the junction conditions (12) and (13).

To obtain the unknown transform coefficient $b_1(s,\ p)$ we form the residuals of the differential equations (9) and (10) and require the orthogonality of the residuals to the coordinate functions (16) and (17):

$$\int_0^{\rho_1}\Big[p(T_w+b_1\varphi_{11}-T_{i1})+s\left(\frac{\rho}{\rho_1}-\frac{\rho^2}{\rho_1^2}\right)(T_w+b_1\varphi_{11}-T_{01})-$$
$$-\frac{a_1}{a}b_1\frac{\partial^2\varphi_{11}}{\partial\rho^2}\Big]\varphi_{11}d\rho+\int_{\rho_1}^1\Big\{p(T_w+b_1\varphi_{12}-T_{i2})+s\Big[\frac{\rho-\rho_1}{1-\rho_1}-$$
$$-\frac{(\rho-\rho_1)^2}{(1-\rho_1)^2}\Big](T_w+b_1\varphi_{12}-T_{01})-\frac{a_2}{a}b_1\frac{\partial^2\varphi_{12}}{\partial\rho^2}\Big\}\varphi_{12}d\rho=0.$$

Upon determining the integrals for the unknown coefficient $b_1(s,\ p)$, we obtain the formula

$$b_1(s,\ p)=\frac{pF_1+sF_2}{pF_3+sF_4+F_5}, \tag{18}$$

where

$$F_1=(T_{i1}-T_w)\int_0^{\rho_1}\varphi_{11}d\rho+(T_{i2}-T_w)\int_{\rho_1}^1\varphi_{12}d\rho;$$

$$F_2 = (T_{01} - T_w)\int_0^{\rho_1}\left(\frac{\rho}{\rho_1} - \frac{\rho^2}{\rho_1^2}\right)\varphi_{11}d\rho + (T_{02} - T_w)\times$$

$$\times\int_{\rho_1}^{1}\left[\frac{\rho-\rho_1}{1-\rho_1} - \frac{(\rho-\rho_1)^2}{(1-\rho_1)^2}\right]\varphi_{12}d\rho;\quad F_3 = \int_0^{\rho_1}\varphi_{11}^2 d\rho + \int_{\rho_1}^{1}\varphi_{12}^2 d\rho;$$

$$F_4 = \int_0^{\rho_1}\left(\frac{\rho}{\rho_1} - \frac{\rho^2}{\rho_1^2}\right)\varphi_{11}^2 d\rho + \int_{\rho_1}^{1}\left[\frac{\rho-\rho_1}{1-\rho_1} - \frac{(\rho-\rho_1)^2}{(1-\rho_1)^2}\right]\varphi_{12}^2 d\rho;$$

$$F_5 = -\frac{a_1}{a}\int_0^{\rho_1}\frac{\partial^2\varphi_{11}}{\partial\rho^2}\varphi_{11}d\rho - \frac{a_2}{a}\int_{\rho_1}^{1}\frac{\partial^2\varphi_{12}}{\partial\rho^2}\varphi_{12}d\rho.$$

In the space of the originals the solution of the problem becomes

$$T_{1i}(\rho, z, \mathrm{Fo}) = \begin{cases} T_w + \dfrac{F_1}{F_3}\exp\left(-\dfrac{F_5}{F_3}\mathrm{Fo}\right)\varphi_{1i}(\rho) & \text{for } z > \dfrac{F_4}{F_3}\mathrm{Fo}, \\ & (i = 1,\ 2) \\ T_w + \dfrac{F_2}{F_4}\exp\left(-\dfrac{F_5}{F_4}z\right)\varphi_{1i}(\rho) & \text{for } z < \dfrac{F_4}{F_3}\mathrm{Fo}. \end{cases} \tag{19}$$

In Eqs. (19) the top equation coincides with the approximate solution of the problem (1)-(8) when the convective terms in Eqs. (1) and (2) are equated to zero (the second terms on the left side of these equations), i.e., with the solution of the nonstationary heat-conduction problem for the contact of two bodies.

In fact, if we put $T_{i1} = T_{i2} = T_i$, $\lambda_1 = \lambda_2$, $a_1 = a_2$, the top equation of Eqs. (19) reduces to the form

$$T_1(\rho, \mathrm{Fo}) = T_w + 1{,}25(T_i - T_w)\exp(-2{,}5\,\mathrm{Fo})\varphi_1(\rho).$$

This expression is in complete agreement with the solution of the nonstationary heat-conduction problem in the first approximation given in [2].

It will be shown later that the bottom equation of Eqs. (19) is in complete agreement with the approximate solution of the corresponding stationary problem, i.e., when the first terms on the left side of Eqs. (1) and (2) are equated to zero.

Thus, for regions of the heat exchanger as yet unperturbed, the heat exchange due to the initial fluid temperatures T_{01} and T_{02} at the channel entrances (for z = 0) takes place as it would in a stationary fluid, i.e., heat transport occurs through heat conduction only.

For heat exchanger regions subjected to the influence of thermal conditions at the channel entrances (fluids, formerly at the channel entrance, having reached these regions) the heat exchange does not depend on the initial conditions T_{i1} and T_{i2} (for Fo = 0). The heat exchange in this case does not depend on the time and is completely determined by the flow of the media, i.e., the problem becomes a stationary one with convective heat transfer along the z axis taken into account.

We carry out the following approximations separately for the nonstationary and the stationary problems [2].

To solve the stationary problem we use the orthogonal method of Kantorovich. In the system of equations (1)-(8) one must equate to zero the first terms on the left side of Eqs. (1) and (2). Following the method of Kantorovich, we seek a solution of the stationary problem in the form

$$T_{ni}(\rho, z) = T_w + \sum_{k=1}^{n} f_k(z)\varphi_{ki}(\rho), \tag{20}$$

where n is the number of the approximations; $f_k(z)$ is an unknown function; $\varphi_{ki}(\rho)$ are coordinate functions.

The coordinate functions of the first approximation are obtained from expressions (16) and (17). Coordinate functions of the following approximations $\varphi_{ki}(\rho)$ $(k = \overline{2, n})$ are obtained in accordance with the expressions $\varphi_{k1}(\rho) = (\rho^2 - \rho_1^2)^2\rho^{2(k-1)}$, $\varphi_{k2}(\rho) = (\rho^2 - 1)^2(\rho^2 - \rho_1^2)^{2(k-1)}$ $(k = \overline{2, n})$.

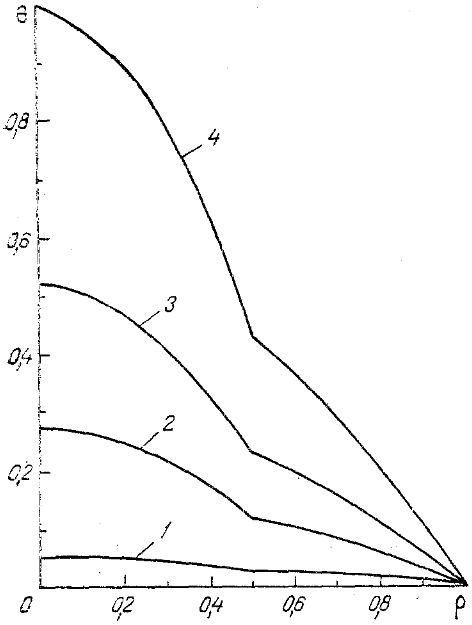

Fig. 1. Variation of the temperature in plane-parallel channels ($z < F_4Fo/F_3$): Curves labeled 1, 2, 3, and 4 are for the z-values 0.1, 0.05, 0.03, and 0.01, respectively.

To find the solution in the first approximation, we form the residual of the differential equations and require that it be orthogonal to the coordinate functions $\varphi_{1i}(\rho)$ $(i=1, 2)$:

$$\left[\frac{\partial}{\partial z}\int_0^{\rho_1}\left(1-\frac{\rho}{\rho_1}\right)\frac{\rho}{\rho_1}T_{11}(\rho, z)-\frac{a_1}{a}\frac{\partial^2 T_{11}(\rho, z)}{\partial\rho^2}\right]\varphi_{11}d\rho+$$
$$+\left[\frac{\partial}{\partial z}\int_{\rho_1}^{1}\left(1-\frac{\rho-\rho_1}{1-\rho_1}\right)\left(\frac{\rho-\rho_1}{1-\rho_1}\right)T_{12}(\rho, z)-\frac{a_2}{a}\frac{\partial^2 T_{12}(\rho, z)}{\partial\rho^2}\right]\varphi_{12}d\rho=0. \quad (21)$$

If in Eq. (21) we replace $T_{11}(\rho, z)$ and $T_{12}(\rho, z)$ by their values from Eq. (20), we find

$$N_1\frac{\partial f_1(z)}{\partial z}+N_2 f_1(z)=0, \quad (22)$$

where

$$N_1=\int_0^{\rho_1}\left(1-\frac{\rho}{\rho_1}\right)\frac{\rho}{\rho_1}\varphi_{11}^2 d\rho+\int_{\rho_1}^{1}\left(1-\frac{\rho-\rho_1}{1-\rho_1}\right)\left(\frac{\rho-\rho_1}{1-\rho_1}\right)\varphi_{12}^2 d\rho;$$

$$N_2=-\frac{a_1}{a}\int_0^{\rho_1}\frac{\partial^2\varphi_{11}}{\partial\rho^2}\varphi_{11}d\rho-\frac{a_2}{a}\int_{\rho_1}^{1}\frac{\partial^2\varphi_{12}}{\partial\rho^2}\varphi_{12}d\rho.$$

The general integral of Eq. (22) is $f_1(z) = C_1\exp(-\nu z)$, where $\nu = N_2/N_1$.

We find the integration constant C_1 from the initial condition (4):

$$\int_0^{\rho_1}(T_w-T_{01}+C_1\varphi_{11})\varphi_{11}d\rho+\int_{\rho_1}^{1}(T_w-T_{02}+C_1\varphi_{12})\varphi_{12}d\rho=0.$$

From this we obtain

$$C_1=\frac{(T_{01}-T_w)\int_0^{\rho_1}\varphi_{11}d\rho+(T_{02}-T_w)\int_{\rho_1}^{1}\varphi_{12}d\rho}{\int_0^{\rho_1}\varphi_{11}^2 d\rho+\int_{\rho_1}^{1}\varphi_{12}^2 d\rho}.$$

The solution of the problem in the first approximation may be described by the expression

$$T_{1i}(\rho, z)=T_w+C_1\exp(-\nu z)\varphi_{1i}(\rho).$$

By way of example, we find the solution of a specific problem for the following initial conditions: $r_1 = 0.008$ m; $r_2 = 0.016$ m; $a_1 = 6 \cdot 10^{-8}$ m²/sec; $a_2 = 12 \cdot 10^{-6}$ m²/sec; $\lambda_1 = 0.11$ W/μm; $\lambda_2 = 0.43$ W/μm; $T_{01} = T_{02} = T_0 > T_W$.

The relative surplus temperature reduces to the form

$$\Theta_{1i}(\rho, z) = \frac{T_{1i}(\rho, z) - T_W}{T_0 - T_W} = C \exp(-\nu z)\varphi_{1i}(\rho), \tag{23}$$

where

$$C = \frac{\int\limits_0^{\rho_1} \varphi_{11} d\rho + \int\limits_{\rho_1}^{1} \varphi_{12} d\rho}{\int\limits_0^{\rho_1} \varphi_{11}^2 d\rho + \int\limits_{\rho_1}^{1} \varphi_{12}^2 d\rho}.$$

For the bottom equation in Eqs. (19) the relative surplus temperature is

$$\Theta_{1i}(\rho, z) = \frac{F_2}{F_4} \exp\left(-\frac{F_5}{F_4} z\right) \varphi_{1i}(\rho) \quad \text{for} \quad z < \frac{F_4}{F_3} \text{Fo}. \tag{24}$$

The calculations made from Eq. (23) are in complete agreement with those made from Eq. (24). The results are shown in Fig. 1 for four different values of z.

The solution of the nonstationary problem (in this case we have, in effect, a nonstationary contact problem of heat conduction for a two-layered plate) is exactly the same as that obtained using the method of Kantorovich. The only difference is that in the expression (20) the $f_k(z)$ is replaced by $f_k(\text{Fo})$ $(k = \overline{1, n})$. The solution of the nonstationary problem obtained by the author using the method of Kantorovich is, in the first approximation, in complete agreement with the top expression in the equations (19).

The approach outlined here can also be applied in solving symmetric problems in cylindrical coordinates (for a tube-within-a-tube type heat exchanger). An expression for the flow rate in the intertube space is given in [2]. There is no change in the expressions for the coordinate functions.

This method can also be used for multi-layered heat exchangers. The coordinate functions for heat exchangers with an arbitrary number of layers may be obtained from the following expressions (for boundary conditions of the form (5) and (8) and for junction conditions of the form (6) and (7)):

$$\varphi_{1i}(\rho) = \sum_{k=0}^{m-i} [1 - H(i + k - m)] \left(\frac{\lambda_m}{\lambda_{m-k}} - \frac{\lambda_m}{\lambda_{m-k-1}}\right)(1 - \rho_{m-k-1}^2) + \frac{\lambda_m}{\lambda_i}(1 - \rho^2) \quad (i = \overline{1, m}); \tag{25}$$

$$\varphi_{ki}(\rho) = (\rho^2 - \rho_i^2)^2 (\rho^2 - \rho_{i-1}^2)^{2[1-H(1-i)]} \rho^{2(k-1)} \quad (i = \overline{1, m}; \quad k = \overline{2, n}). \tag{26}$$

A solution of the form (15), written in terms of the coordinate functions obtained from the equations (25) and (26), satisfies the boundary conditions and the junction conditions for an arbitrary number of heat exchanger layers and for an arbitrary number of approximations.

Media velocity profiles in an arbitrary heat exchanger channel (plane-parallel channels) are determined from the following general expression:

$$W_i(r) = W_{i\text{av}}\left(1 - \frac{\rho - \rho_{i-1}}{\rho_i - \rho_{i-1}}\right)\left(\frac{\rho - \rho_{i-1}}{\rho_i - \rho_{i-1}}\right) \quad (i = \overline{1, m}).$$

For symmetric problems with boundary conditions of the third kind of the form

$$\frac{\partial T_m(1, z, \text{Fo})}{\partial \rho} + \text{Bi}_m [T_m(1, z, \text{Fo}) - T_{\text{ext}}] = 0$$

the expression for the system of coordinate functions in the first approximation is

$$\varphi_{1i}(\rho) = \frac{\text{Bi}_m + 2}{\text{Bi}_m} + \sum_{k=0}^{m-i} [1 - H(i + k - m)] \left(\frac{\lambda_m}{\lambda_{m-k}} - \frac{\lambda_m}{\lambda_{m-k-1}}\right) \rho_{m-k-1}^2 + \frac{\lambda_m}{\lambda_i} \rho^2 \quad (i = \overline{1, m}), \tag{27}$$

where $Bi_m = \alpha_m R/\lambda_m$ is the Biot number; α_m is the heat transfer coefficient between the last (m-th) layer of the heat exchanger and the medium that washes over it; T_{ext} is the temperature of the medium external to the heat exchanger.

Coordinate functions for the successive approximations are obtained from Eq. (26).

In this case, solutions of the form (15) and (20) are to be taken in the following forms:

$$\overline{T}^*_{1i}(\rho,\ s,\ p) = T_{ext} - b_1(s,\ p)\,\varphi_{1i}(\rho)\ \ (i = \overline{1,\ m}), \tag{28}$$

$$T_{ni}(\rho,\ z) = T_{ext} - \sum_{k=1}^{n} f_k(z)\,\varphi_{ki}(\rho)\ \ (i = \overline{1,\ m}). \tag{29}$$

The expressions (28) and (29), with coordinate functions obtained from equations (26) and (27), satisfy the boundary conditions and all the junction conditions. The unknown coefficients $b_1(s,\ p)$ and $f_k(z)$ are to be determined in such a way that the initial differential equations are satisfied in an optimum manner. For this purpose one can use the Bubnov–Galerkin orthogonal method (for determining $b_1(s,\ p)$) and the Kantorovich method (for determining $f_k(z)$ $(k = \overline{1,\ n})$.

It should be pointed out in conclusion that the approach outlined here makes it possible to solve effectively heat exchange junction problems for boundary conditions varying in time and with respect to z, as well as for fluid temperatures at the channel entrances varying with respect to ρ and with respect to the time, and for initial heat-carrier temperatures dependent on the coordinates ρ and z.

NOTATION

T, temperature; T_{I1}, T_{I2}, T_I, initial temperatures; T_w, outer wall temperature; T_{ext}, temperature of external medium; W_{av}, average velocity; x, r, longitudinal and transverse coordinates; τ, time; r_1, $r_2 = R$, distances to inner and outer walls; $\rho = r/R$, dimensionless coordinate; a, smaller of the diffusivity coefficients a_1 and a_2; $Pe = 6RW_{av}/a$, Peclet number; $z = (1/Pe)x/R$, dimensionless coordinate; $Fo = a\tau/R^2$, Fourier number; λ, coefficient of thermal conductivity; m, number of heat exchanger layers; H(η), Heaviside function; η, argument of Heaviside function; α, heat transfer coefficient; $Bi = \alpha R/\lambda$, Biot number.

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THE DYNAMICS OF THE FREEZING OVER OF UNDERGROUND PIPES

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The article suggests a method of calculating the unsteady process of freezing over of an underground pipe transporting a freezing liquid.

Pipeline transport of water, aqueous solutions and suspensions under conditions of low ambient temperatures may be accompanied by their freezing. The formation of an ice layer

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